数理解析研究所講究録
1067 巻 1998 年 135-140

ERROR ESTIMATES OF THE REAL INVERSION FORMULAS OF THE LAPLACE TRANSFORM(abstract)

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INTRODUCTION AND RESULTS

For any $q > 0$, we let L_q^2 be the class of all square integrable functions with respect to the measure $t^{1-2q}dt$ on the half line $(0, \infty)$. Then we consider the Laplace transform

$$[\mathcal{L}F](x) = \int_0^\infty F(t)e^{-xt}dt \quad (x > 0)$$

for $F \in L_q^2$. Then we have

Proposition 1 ([2, 5]). *For any fixed $q > 0$ and for any function $F \in L_q^2$, put $f = \mathcal{L}F$. Then the inversion formula*

$$F(t) = s - \lim_{N\to\infty} \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \qquad (t > 0) \tag{1}$$

is valid, where the limit is taken in the space L_q^2 and the polynomials $P_{N,q}$ are given by the formulas

$$P_{N,q}(\xi) = \sum_{0\le\nu\le n\le N} \frac{(-1)^{\nu+1}\Gamma(2n+2q)}{\nu!(n-\nu)!\Gamma(n+2q+1)\Gamma(n+\nu+2q)}\xi^{n+\nu+2q-1}$$
$$\times \left\{ \frac{2(n+q)}{n+\nu+2q}\xi^2 - \left(\frac{2(n+q)}{n+\nu+2q} + 3n + 2q \right)\xi + n(n+\nu+2q) \right\}.$$

Moreover the series

$$\sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[xf'(x)]|^2 x^{2n+2q-1}dx \tag{2}$$

converges and the truncation error is estimated by the inequality

$$\left\| F(t) - \int_0^\infty f(x)e^{-xt}P_{N,q}(xt)dx \right\|_{L_q^2}^2 \tag{3}$$
$$\le \sum_{n=N+1}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \left|\partial_x^n[xf'(x)]\right|^2 x^{2n+2q-1}dx.$$

Some characteristics of the strong singularity of the polynomials $P_{N,1}(\xi)$ and some effective algorithms for the real inversion formula in Proposition 1 are examined by J. Kajiwara and M. Tsuji [3, 4] and K. Tsuji [6]. Furthermore they gave numerical experiments by using computers.

In connection with the integral in (2) we have

Proposition 2 ([5], Chapter 5). *Let $q > 0$ be arbitrary and let $F \in L_q^2$. For the Laplace transform $\mathcal{L}F = f$, we have the isometrical identity*

$$\int_0^\infty |F(t)|^2 t^{1-2q} dt = \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx. \tag{4}$$

Moreover the image $f = \mathcal{L}F$ belongs to the Bergman-Selberg space $H_q(R^+)$ on the right half complex plane $R^+ = \{Re\ z > 0\}$ admitting the reproducing kernel

$$K_q(z, \overline{u}) = \frac{\Gamma(2q)}{(z+\overline{u})^{2q}}$$

and comprising analytic functions on R^+. For $q > \frac{1}{2}$, we can characterize

$$\begin{aligned} H_q(R^+) = &\{f : f \quad \text{analytic on} \quad R^+, \\ &\frac{1}{\Gamma(2q-1)\pi} \iint_{R^+} |f(z)|^2 (2x)^{2q-2} dx dy < \infty\} \end{aligned}$$

and for $q = \frac{1}{2}$

$$\begin{aligned} H_{\frac{1}{2}}(R^+) = &\{f : f \quad \text{analytic on} \quad R^+, \\ &\lim_{x \to +0} \frac{1}{2\pi} \int_{-\infty}^\infty |f(x+iy)|^2 dy < \infty\}. \end{aligned}$$

Moreover for any $q > 0$, we have the representation of the norm in $H_q(R^+)$

$$\|f\|^2_{H_q(R^+)} = \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n (xf'(x))|^2 x^{2n+2q-1} dx. \tag{5}$$

Now we can state our main results.

Theorem 1. *We assume that*

$$\max\left(\frac{1}{2}, 2q-1\right) < \alpha < 1, \tag{6}$$

and

$$\alpha \leq \beta < q + \frac{\alpha}{2}. \tag{7}$$

If $f \in H_q(R^+)$ *and*

$$f(z)z^{\beta} \in H_{q+\frac{\alpha}{2}-\beta}(R^+), \tag{8}$$

then the following error estimate holds

$$\left| F(t) - \int_0^{\infty} f(x)e^{-xt} P_{N,q}(xt)dx \right| = t^{q-1+\frac{\alpha}{2}} o\left(N^{\frac{1-2\alpha}{4}}\right) \tag{9}$$

as $N \to \infty$.

Next we give a sufficient condition for F whose Laplace transform satisfies (8).

Theorem 2. *Let us assume (7). We further assume*

$$q + \frac{\alpha}{2} > 1. \tag{10}$$

If

$$F \in C^2[0, \infty), \tag{11}$$

$$F(0) = F'(0) = 0, \tag{12}$$

and

$$F'(t) = O(t^{-\delta}), \quad t > 0 \tag{13}$$

for

$$2 - q - \frac{\alpha}{2} < \delta < 1, \tag{14}$$

then (8) holds.

Note that from (12) and (13)

$$\lim_{t \to \infty} e^{-xt} F(t) = \lim_{t \to \infty} e^{-xt} F'(t) = 0, \quad x > 0. \tag{15}$$

Finally, we characterize F whose Laplace transform satisfies (8).

Theorem 3. *If $f = \mathcal{L}F$ satisfies (8), then there exists $h \in L^2_{q+\frac{\alpha}{2}-\beta}$ such that (7) is true and*

$$F(t) = \int_0^t h(x)(t-x)^{\beta-1}dx. \tag{16}$$

A real inversion formula for the Laplace transform is known (eg. Widder [7], page 386), which is different from ours. However it seems that no error estimates in the truncation are known.

PRELIMINARIES

First we shall give

Lemma. *If $f \in C^\infty(0,\infty)$ and*

$$I_{q,\alpha}(f) := \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial_x^n[xf'(x)]|^2 x^{2n+2q-1+\alpha} dx < \infty, \tag{17}$$

for fixed

$$\max\left(\frac{1}{2}, 2q-1\right) < \alpha, \tag{18}$$

then

$$\begin{aligned} &\left| \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \right. \\ &\qquad \left. \times \int_0^\infty \partial_x^n[xf'(x)]\partial_x^n(x\partial_x(e^{-tx}))x^{2n+2q-1}dx \right| \\ &= t^{\frac{\alpha-2q}{2}} o(N^{\frac{1-2\alpha}{4}}), \end{aligned} \tag{19}$$

as $N \to \infty$.

CONCLUDING REMARKS

(1) The conditions (12) and (13) are not essential if we know $F(0)$ and $F'(0)$, and we can assume that

$$|F(t)|, |F'(t)| \leq O(e^{kt}) \quad \text{for} \quad t > 0 \quad \text{with} \quad k > 0. \tag{20}$$

In fact, we set

$$\tilde{F}(t) = (F(t) - F(0) - F'(0)t)e^{-2kt}, \quad t > 0. \tag{21}$$

Then $\tilde{F}$ satisfies (12) and (13).

On the other hand,

$$(\mathcal{L}\tilde{F})(z) = f(z+2k) - \frac{F(0)}{z+2k} - \frac{F'(0)}{(z+2k)^2}. \tag{22}$$

Thus we first apply Theorems 1 and 2 to this function (22) so that we can obtain approximations $\tilde{F}_N(t)$ for $\tilde{F}(t)$:

$$|\tilde{F}(t) - \tilde{F}_N(t)| = t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}). \tag{23}$$

We set

$$\hat{F}_N(t) = \tilde{F}_N(t)e^{2kt} + F(0) + F'(0)t, \quad \text{for} \quad t > 0. \tag{24}$$

Then we have

$$|F(t) - \hat{F}_N(t)| = e^{2kt}|\tilde{F}(t) - \tilde{F}_N(t)| = e^{2kt} t^{q-1+\frac{\alpha}{2}} o(N^{\frac{1-2\alpha}{4}}). \tag{25}$$

Thus we can obtain error estimates in any finite interval in t, which however breakes as $t \to \infty$.

(2) Since a typical member of the Bergman-Selberg space $H_q(R^+)$ is the reproducing kernel $K_q(z, \overline{u})$, we see that typical functions f satisfying (17) are given by

$$f(z) = \frac{z^{-\beta}}{(z+\overline{u})^{2q+\alpha-2\beta}}, \quad \mathrm{Re}\, u > 0 \tag{26}$$

for α and β satisfying (7). From the identities (16) and

$$K_{q+\frac{\alpha}{2}-\beta}(z, \overline{u}) = \int_0^\infty e^{-tz} e^{-t\overline{u}} t^{2q+\alpha-2\beta-1} dt,$$

we see that the Laplace transform of the functions

$$\int_0^t e^{-x\overline{u}} x^{2q+\alpha-2\beta-1} (t-x)^{\beta-1} dx, \quad \mathrm{Re}\, u > 0,\ \beta > 1 \tag{27}$$

satisfies the property (17).

(3) As functions F where $f = \mathcal{L}F$ satisfies the conditions in Theorem 1, we consider Dirichlet series

$$F(t) = \sum_{k=1}^{\infty} C_k t^{\gamma-1} e^{-a_k t} \quad (a_k > 0, \gamma \geq 1), \tag{28}$$

where

$$\sum_{k=1}^{\infty} |C_k| a_k^{q-\gamma} < \infty, \quad \sum_{k=1}^{\infty} |C_k| a_k^{q+\frac{\alpha}{2}-\gamma} < \infty, \quad \gamma > q + \frac{\alpha}{2} > 0. \tag{29}$$

Then $F \in L_q^2$ and $f = \mathcal{L}F$ satisfies (8) for β satisfying (7).

ACKNOWLEDGEMENT

This research was partially supported by the Japanese Ministry of Education, Science, Sports and Culture; Grant-in-Aid Scientific Research, Kiban Kenkyuu (A)(1), 10304009.

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